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Exploring error estimates of Newton-Cotes quadrature rules across diverse function classes

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Abstract

This in-depth study looks at symmetric four-point Newton-Cotes-type inequalities with a focus on error estimates for numerical integration. The precision of these estimates is explored across various classes of functions, including those with bounded variation, bounded derivatives, Lipschitzian derivatives, convex derivatives, and others. The research synthesizes and extends existing knowledge, providing a nuanced understanding of how error bounds depend on the characteristics of integrated functions. Through a systematic review of seminal works, the study contributes to the practical application of numerical integration techniques, offering insight for researchers and practitioners to make informed choices based on the specific features of the functions involved.

Mathematics Subject Classification: 26D10; 26D15; 26A51

Keywords: Newton-Cotes inequalities; Functions of bounded variation; Lipschitzian functions; Bounded functions; Convex functions

1 Introduction

Numerical integration techniques, particularly Newton–Cotes quadrature rules, have long been established as indispensable tools for approximating definite integrals. While these methods are well-known for their simplicity and ease of implementation, a crucial aspect of their utility lies in the ability to provide accurate estimates of the associated errors.

The precision of these estimates, of course, hinges on the nature of the functions being integrated. Understanding the dependence of error bounds on the characteristics of the integrated function is paramount to ensuring the reliability of numerical results. In this context, extensive research has been conducted to obtain error estimates tailored to various classes of functions. For bounded functions [2, 19], Lipschitzian functions [9, 10, 12, 20, 27, 29], functions of bounded variation [8, 11, 14–16, 28], convex functions [5, 21, 23–26, 31], among others.

Several seminal works have paved the way in this direction, offering invaluable contributions to the field of numerical analysis. These efforts have not only enhanced our theoretical understanding, but have also provided practitioners with practical guidelines for selecting the most suitable numerical integration techniques based on the nature of the functions involved.

In [14], Dragomir established the following result related to companion of Ostrowski-type inequalities for functions of bounded variations

$$\left|\frac{f(x)+f(\varrho_1+\varrho_2-x)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right| \leq \left(\frac{1}{4}+\frac{\left|\frac{3\varrho_1+\varrho_2}{4}-x\right|}{(\varrho_2-\varrho_1)}\right)\bigvee_{\varrho_1}^{\varrho_2}(f),$$

where $x\in\left[\varrho_1,\frac{\varrho_1+\varrho_2}{2}\right]$.

The Simpson-type inequality for functions of bounded variations was provided in [8] as follows:

$$\left|\frac{1}{6}\left(f(\varrho_1)+4f\left(\frac{\varrho_1+\varrho_2}{2}\right)+f(\varrho_2)\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right| \leq \frac{1}{3}\bigvee_{\varrho_1}^{b}(f).$$

Moreover, Alomari [3] provided the Simpson second formula inequality for the same class of functions as follows:

$$\left|\frac{1}{8}\left(f(\varrho_1)+3f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+3f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+f(\varrho_2)\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right| \leq \frac{5}{24}\bigvee_{\varrho_1}^{\varrho_2}(f).$$

Alomari et al. provided a companion of Ostrowski's inequalities for differentiable convex functions in [5] as follow:

$$\begin{aligned}&\left|\frac{f(x)+f(\varrho_1+\varrho_2-x)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|\\ &\leq \frac{1}{\varrho_2-\varrho_1}\left(\frac{1}{6}(x-\varrho_1)^2\left(\left|f'(\varrho_1)\right|+\left|f'(\varrho_2)\right|\right)\right.\\ &\quad\left.+\left(\frac{1}{3}(x-\varrho_1)^2+\frac{1}{2}\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2\right)\left(\left|f'(x)\right|+\left|f'(\varrho_1+\varrho_2-x)\right|\right)\right).\end{aligned}$$

Moreover, Kashuri et al. [23] established the Simpson-type inequalities via convexity:

$$\begin{aligned}&\left|\frac{1}{6}\left(f(\varrho_1)+4f\left(\frac{\varrho_1+\varrho_2}{2}\right)+f(\varrho_2)\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|\\ &\leq \frac{\varrho_2-\varrho_1}{324}\left(8\left|f'(\varrho_1)\right|+29\left|f'\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|+8\left|f'(\varrho_2)\right|\right).\end{aligned}$$

Furthermore, the authors in [26] gave the Simpson second formula for the same class of functions

$$\begin{aligned}&\left|\frac{1}{8}\left(f(\varrho_1)+3f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+3f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+f(\varrho_2)\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|\\ &\leq \frac{\varrho_2-\varrho_1}{13{,}824}\left(157\left(\left|f'(\varrho_1)\right|+\left|f'(\varrho_2)\right|\right)+443\left(\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|+\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|\right)\right).\end{aligned}$$

In this paper, we investigate a symmetric bi-parametric formula involving four points. We commence by introducing a novel identity related to the aforementioned formula, from which we establish numerous inequalities for functions whose first derivative belongs to various classes, such as bounded variation, Lipschitzian, convex, among others. The obtained results represent a generalization and extension of several previous works, opening up new avenues for further research in this direction.

2 Preliminaries

Definition 2.1 ([21]) Consider a partition $\mathcal{P} : \varrho_1 = \xi_0 < \xi_1 < \cdots < \xi_n = \varrho_2$ of the interval $[\varrho_1, \varrho_2]$, and define $\Delta f(\xi_i) = f(\xi_{i+1}) - f(\xi_i)$. A function $f(x)$ is considered of bounded variation if the sum $\sum\limits_{i=1}^{n} \Delta f(\xi_i)$ is bounded for all such partitions.

Definition 2.2 ([21]) Consider a function f defined on the interval $[\varrho_1, \varrho_2]$ with bounded variation. Let $P([\varrho_1, \varrho_2])$ represent the family of partitions of $[\varrho_1, \varrho_2]$, and $\sum(\mathcal{P})$ denote the sum $\sum\limits_{i=1}^{n} \left|\Delta f(\xi_i)\right|$ corresponding to the partition $\mathcal{P}$ of $[\varrho_1, \varrho_2]$. The total variation of f on $[\varrho_1, \varrho_2]$, denoted as $\bigvee_{\varrho_1}^{\varrho_2}(f)$, is given by $\bigvee_{\varrho_1}^{\varrho_2}(f) = \sum(\mathcal{P})$ with $\mathcal{P} \in P([\varrho_1, \varrho_2])$.

Lemma 2.3 ([6]) *Consider functions $g, f : [\varrho_1, \varrho_2] \to \mathbb{R}$, where g is continuous on $[\varrho_1, \varrho_2]$ and f has bounded variation on $[\varrho_1, \varrho_2]$. Then, the integral $\int_{\varrho_1}^{\varrho_2} g(t)\, df(t)$ exists and satisfies the inequality*

$$\left|\int\limits_{\varrho_1}^{\varrho_2} g(t)\, df(t)\right| \leq \sup_{t\in[\varrho_1,\varrho_2]} \left|g(t)\right| \bigvee_{\varrho_1}^{\varrho_2}(f). \tag{2.1}$$

Lemma 2.4 ([6]) *Consider functions $g, f : [\varrho_1, \varrho_2] \to \mathbb{R}$, where g is Riemann integrable on $[\varrho_1, \varrho_2]$, and f is an L-Lipschitzian function. Then, the following inequality holds*:

$$\left|\int_{\varrho_1}^{\varrho_2} g(t)\, df(t)\right| \leq L \int_{\varrho_1}^{\varrho_2} \left|g(t)\right| dt. \tag{2.2}$$

3 Main results

Let us commence by introducing the following integral identity, which will serve as a primary tool in establishing our results.

Lemma 3.1 *Consider a function $f : I \subset \mathbb{R} \to \mathbb{R}$ that is differentiable on the interior I°, where $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2$. Assume that the derivative f' is integrable on the interval $[\varrho_1, \varrho_2]$, denoted as $f'^1[\varrho_1, \varrho_2]$. Then, the following equality holds for any real number λ in the interval $[0, 1]$ and any x in the interval $[\varrho_1, \frac{\varrho_1+\varrho_2}{2}]$*:

$$Q_\lambda\left(\varrho_1, x, \varrho_2; f\right) - \frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2} f(u)\, du = \frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2} K(t,x)\, df(t), \tag{3.1}$$

where

$$Q_\lambda\left(\varrho_1, x, \varrho_2; f\right) = \frac{\lambda(x-\varrho_1)}{\varrho_2-\varrho_1} f(\varrho_1) + \frac{2(1-\lambda)(x-\varrho_1)+\varrho_1+\varrho_2-2x}{2(\varrho_2-\varrho_1)} f(x) \tag{3.2}$$

$$+\tfrac{2(1-\lambda)(x-\varrho_1)+\varrho_1+\varrho_2-2x}{2(\varrho_2-\varrho_1)}f(\varrho_1+\varrho_2-x)+\tfrac{\lambda(x-\varrho_1)}{\varrho_2-\varrho_1}f(\varrho_2)$$

and

$$K(t,x)=\begin{cases} t-\varrho_1-\lambda(x-\varrho_1) & \text{if} \quad t\in[\varrho_1,x],\\ t-\frac{\varrho_1+\varrho_2}{2} & \text{if} \quad t\in[x,\varrho_1+\varrho_2-x],\\ t-\varrho_2+\lambda(x-\varrho_1) & \text{if} \quad t\in[\varrho_1+\varrho_2-x,\varrho_2].\end{cases} \tag{3.3}$$

Proof Clearly, $t-\varrho_1-\lambda(x-\varrho_1)$, $t-\frac{\varrho_1+\varrho_2}{2}$ and $t-\varrho_2+\lambda(x-\varrho_1)$ are continuous on $[\varrho_1,x]$, $[x,\varrho_1+\varrho_2-x]$ and $[\varrho_1+\varrho_2-x,\varrho_2]$, respectively, and f is of bounded variation, then all the Riemann–Stieltjes integrals exist.

Using the integration by parts formula for Riemann–Stieltjes integral, we conclude

$$\begin{aligned}\int_{\varrho_1}^{\varrho_2}K(t,x)\,df(t)&=\int_{\varrho_1}^{x}(t-\varrho_1-\lambda(x-\varrho_1))\,df(t)+\int_{x}^{\varrho_1+\varrho_2-x}\left(t-\tfrac{\varrho_1+\varrho_2}{2}\right)df(t)\\
&\quad+\int_{\varrho_1+\varrho_2-x}^{\varrho_2}(t-\varrho_2+\lambda(x-\varrho_1))\,df(t)\\
&=(t-\varrho_1-\lambda(x-\varrho_1))f(t)\Big|_{t=a}^{t=x}+\left(t-\tfrac{\varrho_1+\varrho_2}{2}\right)f(t)\Big|_{t=x}^{t=\varrho_1+\varrho_2-x}\\
&\quad+(t-\varrho_2+\lambda(x-\varrho_1))f(t)\Big|_{t=\varrho_1+\varrho_2-x}^{t=b}-\int_{\varrho_1}^{\varrho_2}f(u)\,du\\
&=(x-\varrho_1-\lambda(x-\varrho_1))f(x)-(\varrho_1-\varrho_1-\lambda(x-\varrho_1))f(\varrho_1)\\
&\quad+\left(\varrho_1+\varrho_2-x-\tfrac{\varrho_1+\varrho_2}{2}\right)f(\varrho_1+\varrho_2-x)-\left(x-\tfrac{\varrho_1+\varrho_2}{2}\right)f(x)\\
&\quad+(\varrho_2-\varrho_2+\lambda(x-\varrho_1))f(\varrho_2)\\
&\quad-(\varrho_1+\varrho_2-x-\varrho_2+\lambda(x-\varrho_1))f(\varrho_1+\varrho_2-x)-\int_{\varrho_1}^{\varrho_2}f(u)\,du\\
&=\lambda(x-\varrho_1)f(\varrho_1)+\tfrac{2(1-\lambda)(x-\varrho_1)+\varrho_1+\varrho_2-2x}{2}f(x)\\
&\quad+\tfrac{2(1-\lambda)(x-\varrho_1)+\varrho_1+\varrho_2-2x}{2}f(\varrho_1+\varrho_2-x)+\lambda(x-\varrho_1)f(\varrho_2)\\
&\quad-\int_{\varrho_1}^{\varrho_2}f(u)\,du.\end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{1}{\varrho_2-\varrho_1}$, we get the result. The proof is completed. □

Theorem 3.2 *Let $f:[\varrho_1,\varrho_2]\to\mathbb{R}$ be a mapping of bounded variation. Then we have*

$$\left|Q_\lambda(\varrho_1,x,\varrho_2;f)-\tfrac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)\,du\right|\le\tfrac{(x-\varrho_1)|1-2\lambda|+\varrho_2-x+|(2+|1-2\lambda|)(x-\varrho_1)-(\varrho_2-x)|}{4(\varrho_2-\varrho_1)}\bigvee_{\varrho_1}^{\varrho_2}(f),$$

where $\lambda\in[0,1]$, $x\in\left[\varrho_1,\frac{\varrho_1+\varrho_2}{2}\right]$ and $Q_\lambda(\varrho_1,x,\varrho_2;f)$ is defined as (3.2).

Proof Taking the absolute value in (3.1) and using (2.1), we get

$$
\begin{aligned}
&\left| Q_{\lambda}\left(\varrho_1, x, \varrho_2; f\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \\
&\leq \frac{1}{\varrho_2-\varrho_1}\left(\left| \int_{\varrho_1}^{x} \left(t-\varrho_1-\lambda\left(x-\varrho_1\right)\right) df(t) \right| + \left| \int_{x}^{\varrho_1+\varrho_2-x} \left(t-\frac{\varrho_1+\varrho_2}{2}\right) df(t) \right| \right. \\
&\quad \left. + \left| \int_{\varrho_1+\varrho_2-x}^{\varrho_2} \left(t-\varrho_2+\lambda\left(x-\varrho_1\right)\right) df(t) \right| \right) \\
&\leq \frac{1}{\varrho_2-\varrho_1}\left(\sup_{t\in[\varrho_1,x]} \left|t-\varrho_1-\lambda\left(x-\varrho_1\right)\right| \bigvee_{\varrho_1}^{x}(f) + \sup_{t\in[\varrho_1,\varrho_1+\varrho_2-x]} \left|t-\frac{\varrho_1+\varrho_2}{2}\right| \bigvee_{x}^{\varrho_1+\varrho_2-x}(f) \right. \\
&\quad \left. + \sup_{t\in[\varrho_1+\varrho_2-x,\varrho_2]} \left|t-\varrho_2+\lambda\left(x-\varrho_1\right)\right| \bigvee_{\varrho_1+\varrho_2-x}^{\varrho_2}(f) \right) \\
&= \frac{1}{\varrho_2-\varrho_1}\left(\max\left\{(1-\lambda)\left(x-\varrho_1\right), \lambda\left(x-\varrho_1\right)\right\} \bigvee_{\varrho_1}^{x}(f) + \left(\frac{\varrho_1+\varrho_2}{2}-x\right) \bigvee_{x}^{\varrho_1+\varrho_2-x}(f) \right. \\
&\quad \left. + \max\left\{(1-\lambda)\left(x-\varrho_1\right), \lambda\left(x-\varrho_1\right)\right\} \bigvee_{\varrho_1+\varrho_2-x}^{x}(f) \right) \\
&= \frac{1}{\varrho_2-\varrho_1}\left(\left(x-\varrho_1\right)\frac{1+|1-2\lambda|}{2} \bigvee_{\varrho_1}^{x}(f) + \left(\frac{\varrho_1+\varrho_2}{2}-x\right) \bigvee_{x}^{\varrho_1+\varrho_2-x}(f) + \left(x-\varrho_1\right)\frac{1+|1-2\lambda|}{2} \bigvee_{\varrho_1+\varrho_2-x}^{x}(f) \right) \\
&\leq \frac{1}{\varrho_2-\varrho_1} \max_{x\in\left[\varrho_1,\frac{\varrho_1+\varrho_2}{2}\right]} \left\{ \left(x-\varrho_1\right)\frac{1+|1-2\lambda|}{2}, \frac{\varrho_1+\varrho_2}{2}-x \right\} \left[\bigvee_{\varrho_1}^{x}(f) + \bigvee_{x}^{\varrho_1+\varrho_2-x}(f) + \bigvee_{\varrho_1+\varrho_2-x}^{x}(f) \right] \\
&\leq \frac{1}{\varrho_2-\varrho_1} \max_{x\in\left[\varrho_1,\frac{\varrho_1+\varrho_2}{2}\right]} \left\{ \left(x-\varrho_1\right)\frac{1+|1-2\lambda|}{2}, \frac{\varrho_1+\varrho_2}{2}-x \right\} \bigvee_{\varrho_1}^{\varrho_2}(f) \\
&= \frac{(x-\varrho_1)|1-2\lambda|+\varrho_2-x+|(2+|1-2\lambda|)(x-\varrho_1)-(\varrho_2-x)|}{4(\varrho_2-\varrho_1)} \bigvee_{\varrho_1}^{\varrho_2}(f).
\end{aligned}
$$

The proof is completed. □

Remark 3.3 Theorem 3.2 will be reduced to

- Theorem 4 from [14], if we take $\lambda = 0$,
- Corollary 1 from [14], if we take $x = \varrho_1$.

Corollary 3.4 *In Theorem* 3.2, *if we take* $x = \frac{\varrho_1+\varrho_2}{2}$, *we get the following parametrized three-point type inequalities*

$$
\left| \frac{\lambda}{2} f\left(\varrho_1\right) + (1-\lambda) f\left(\frac{\varrho_1+\varrho_2}{2}\right) + \frac{\lambda}{2} f\left(\varrho_2\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \leq \frac{1+|1-2\lambda|}{4} \bigvee_{\varrho_1}^{\varrho_2}(f).
$$

Remark 3.5 Corollary 3.4 will be reduced to

- Theorem 2.1 from [11], if we take $\lambda = 0$,
- Theorem 2.1 from [8], if we take $\lambda = \frac{1}{3}$,

- Theorem 2.1 from [13], if we take $\lambda = 1$.
- Corollary 3 (item 3) from [3], if we take $\lambda = \frac{1}{2}$.

Corollary 3.6 *In Theorem 3.2, if we take $x = \frac{2\varrho_1+\varrho_2}{3}$, we get the following parametrized four-point type inequalities*

$$\left|\frac{\lambda}{3}f(\varrho_1) + \frac{3-2\lambda}{6}f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + \frac{3-2\lambda}{6}f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + \frac{\lambda}{3}f(\varrho_2) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{1+|1-2\lambda|}{6}\bigvee_{\varrho_1}^{\varrho_2}(f).$$

Remark 3.7 By taking $\lambda = \frac{3}{8}$ in Corollary 3.6, we retrieve Corollary 3 from [3].

Corollary 3.8 *In Corollary 3.6, if we take $\lambda = \frac{39}{80}$, we get the following corrected Simpson second formula inequality*

$$\left|\frac{1}{80}\left(13f(\varrho_1) + 27f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + 27f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + 13f(\varrho_2)\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{41}{240}\bigvee_{\varrho_1}^{\varrho_2}(f).$$

Theorem 3.9 *Let $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[\varrho_1, \varrho_2]$. Then we have*

$$\left|Q_\lambda\left(\varrho_1, x, \varrho_2; f\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{L}{\varrho_2-\varrho_1}\left(\left(\lambda^2 + (1-\lambda)^2\right)(x-\varrho_1)^2 + \left(\frac{\varrho_1+\varrho_2}{2} - x\right)^2\right),$$

where $\lambda \in [0,1]$, $x \in \left[\varrho_1, \frac{\varrho_1+\varrho_2}{2}\right]$ and $Q_\lambda\left(\varrho_1, x, \varrho_2; f\right)$ is defined as (3.2).

Proof Using the absolute value in (3.1) and using (2.2), we get

$$\left|Q_\lambda\left(\varrho_1, x, \varrho_2; f\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{L}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{x} |t-\varrho_1-\lambda(x-\varrho_1)|\,dt + \int_{x}^{\varrho_1+\varrho_2-x}\left|t-\frac{\varrho_1+\varrho_2}{2}\right|dt\right.$$
$$\left.+\int_{\varrho_1+\varrho_2-x}^{\varrho_2} |t-\varrho_2+\lambda(x-\varrho_1)|\,dt\right)$$
$$= \frac{L}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{\varrho_1+\lambda(x-\varrho_1)} (\varrho_1+\lambda(x-\varrho_1)-t)\,dt + \int_{\varrho_1+\lambda(x-\varrho_1)}^{x} (t-\varrho_1-\lambda(x-\varrho_1))\,dt\right.$$

$$+\int_{x}^{\frac{\varrho_1+\varrho_2}{2}} \left(\frac{\varrho_1+\varrho_2}{2}-t\right) dt$$

$$+\int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_1+\varrho_2-x} \left(t-\frac{\varrho_1+\varrho_2}{2}\right) dt$$

$$\left.+\int_{\varrho_1+\varrho_2-x}^{\varrho_2-\lambda(x-\varrho_1)} \left(\varrho_2-\lambda\left(x-\varrho_1\right)-t\right) dt + \int_{\varrho_2-\lambda(x-\varrho_1)}^{\varrho_2} \left(t-\varrho_2+\lambda\left(x-\varrho_1\right)\right) dt\right)$$

$$= \frac{L}{2(\varrho_2-\varrho_1)} \left(\lambda^2\left(x-\varrho_1\right)^2 + (1-\lambda)^2\left(x-\varrho_1\right)^2 + \left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2\right.$$

$$\left. + \left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2 + (1-\lambda)^2\left(x-\varrho_1\right)^2 + \lambda^2\left(x-\varrho_1\right)^2\right)$$

$$= \frac{L}{(\varrho_2-\varrho_1)} \left(\left(\lambda^2+(1-\lambda)^2\right)\left(x-\varrho_1\right)^2 + \left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2\right).$$

This completes the proof. □

Corollary 3.10 *In Theorem 3.9, if we take* $\lambda = 0$, *we get*

$$\left|\frac{f(x)+f(\varrho_1+\varrho_2-x)}{2} - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\, du\right| \le \frac{L}{\varrho_2-\varrho_1}\left(\left(x-\varrho_1\right)^2 + \left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2\right),$$

which is the same result obtained in inequality (3.10) *from* [4].

Remark 3.11 Theorem 3.9 will be reduced to

- Theorem 2.1 from [9], if we take $x = \varrho_1$,
- Corollary 4 from [29], if we take $x = \frac{\varrho_1+\varrho_2}{2}$. Moreover, we retrieve
  1. Theorem 2.1 from [12], for $\lambda = 0$,
  2. Theorem 2 from [18], for $\lambda = \frac{1}{3}$,
  3. Remark 2 from [29], for $\lambda = \frac{1}{2}$,
  4. Corollary 3.3 from [4], for $\lambda = 1$.

Corollary 3.12 *In Theorem 3.9, if we take* $x = \frac{2\varrho_1+\varrho_2}{3}$, *we get the following parametrized four-point type inequalities*

$$\left|\frac{\lambda}{3}f\left(\varrho_1\right) + \frac{3-2\lambda}{6}f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + \frac{3-2\lambda}{6}f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + \frac{\lambda}{3}f\left(\varrho_2\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\, du\right|$$

$$\le \frac{\varrho_2-\varrho_1}{36}\left(1+4\left(\lambda^2+(1-\lambda)^2\right)\right)L.$$

Remark 3.13 By taking $\lambda = \frac{3}{8}$ in Corollary 3.12, we retrieve Theorem 2 from [21].

Corollary 3.14 *In Corollary 3.12, if we take* $\lambda = \frac{39}{80}$, *we get*

$$\left|\frac{1}{80}\left(13f(\varrho_1) + 27f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + 27f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + 13f(\varrho_2)\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{2401(\varrho_2-\varrho_1)}{28{,}800}L.$$

Theorem 3.15 *Let* $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ *be an absolutely continuous mapping on* $[\varrho_1, \varrho_2]$. *If* $f' \in L^p[\varrho_1, \varrho_2]$, *then we have*

$$\left|Q_\lambda(\varrho_1, x, \varrho_2; f) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\left(\frac{2\left(\lambda^{q+1}+(1-\lambda)^{q+1}\right)(x-\varrho_1)^{q+1}+2\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^{q+1}}{q+1}\right)^{\frac{1}{q}}\left\|f'\right\|_p,$$

where $\lambda \in [0,1]$, $x \in \left[\varrho_1, \frac{\varrho_1+\varrho_2}{2}\right]$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $Q_\lambda(\varrho_1, x, \varrho_2; f)$ *is defined as* (3.2).

Proof Using the absolute value in (3.1) and Hölder's inequality

$$\left|Q_\lambda(\varrho_1, x, \varrho_2; f) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} |K(t,x)|\left|f'(t)\right|dt$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{\varrho_2} |K(t,x)|^q\,dt\right)^{\frac{1}{q}}\left(\int_{\varrho_1}^{\varrho_2} \left|f'(t)\right|^p dt\right)^{\frac{1}{p}}$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{x} |t-\varrho_1-\lambda(x-\varrho_1)|^q\,dt + \int_{x}^{\varrho_1+\varrho_2-x}\left|t-\frac{\varrho_1+\varrho_2}{2}\right|^q dt\right.$$
$$\left.+\int_{\varrho_1+\varrho_2-x}^{\varrho_2} |t-\varrho_2+\lambda(x-\varrho_1)|^q\,dt\right)^{\frac{1}{q}}\left\|f'\right\|_p$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{x} |t-\varrho_1-\lambda(x-\varrho_1)|^q\,dt\right.$$
$$\left.+\int_{x}^{\varrho_1+\varrho_2-x}\left|t-\frac{\varrho_1+\varrho_2}{2}\right|^q dt + \int_{\varrho_1+\varrho_2-x}^{\varrho_2} |t-\varrho_2+\lambda(x-\varrho_1)|^q\,dt\right)^{\frac{1}{q}}\left\|f'\right\|_p$$
$$= \frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{\varrho_1+\lambda(x-\varrho_1)} (\varrho_1+\lambda(x-\varrho_1)-t)^q\,dt\right.$$

$$
\begin{aligned}
&+\int_{\varrho_1+\lambda(x-\varrho_1)}^{x}\left(t-\varrho_1-\lambda\left(x-\varrho_1\right)\right)^q dt+\int_{x}^{\frac{\varrho_1+\varrho_2}{2}}\left(\tfrac{\varrho_1+\varrho_2}{2}-t\right)^q dt\\
&+\int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_1+\varrho_2-x}\left(t-\tfrac{\varrho_1+\varrho_2}{2}\right)^q dt\\
&+\left.\int_{\varrho_1+\varrho_2-x}^{\varrho_2-\lambda(x-\varrho_1)}\left(\varrho_2-\lambda\left(x-\varrho_1\right)-t\right)^q dt+\int_{\varrho_2-\lambda(x-\varrho_1)}^{\varrho_2}\left(t-\varrho_2+\lambda\left(x-\varrho_1\right)\right)^q dt\right)^{\frac{1}{q}}\left\|f'\right\|_p\\
&=\tfrac{1}{\varrho_2-\varrho_1}\left(\tfrac{2\left(\lambda^{q+1}+(1-\lambda)^{q+1}\right)(x-\varrho_1)^{q+1}+2\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^{q+1}}{q+1}\right)^{\frac{1}{q}}\left\|f'\right\|_p .
\end{aligned}
$$

Thus, the proof is completed. □

Remark 3.16 Theorem 3.15 will be reduced to

- Inequality 5.6 of Corollary 5.2 from [4], if we take $\lambda = 0$,
- Inequality 3.9 of Corollary 3.3 from [4], if we take $x = \varrho_1$.

Corollary 3.17 *In Theorem 3.15, if we take $x = \frac{\varrho_1+\varrho_2}{2}$, we get the following parametrized three-point type inequalities*

$$
\begin{aligned}
&\left|\tfrac{\lambda}{2}f\left(\varrho_1\right)+(1-\lambda)f\left(\tfrac{\varrho_1+\varrho_2}{2}\right)+\tfrac{\lambda}{2}f\left(\varrho_2\right)-\tfrac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)\,du\right|\\
&\leq\tfrac{(\varrho_2-\varrho_1)^{\frac{1}{q}}}{2}\left(\tfrac{\lambda^{q+1}+(1-\lambda)^{q+1}}{q+1}\right)^{\frac{1}{q}}\left\|f'\right\|_p .
\end{aligned}
$$

Remark 3.18 Corollary 3.17 will be reduced to

- Inequality 5.3 of Corollary 5.2 from [4], if we take $\lambda = 0$,
- Theorem 2.1 from [7] (see also Theorem 3 from [18]), if we take $\lambda = \frac{1}{3}$,
- Inequality 5.4 of Corollary 5.2 from [4], if we take $\lambda = 1$.

Corollary 3.19 *In Corollary 3.17, if we take $\lambda = \frac{1}{2}$, we get*

$$
\left|\tfrac{1}{4}\left(f\left(\varrho_1\right)+2f\left(\tfrac{\varrho_1+\varrho_2}{2}\right)+f\left(\varrho_2\right)\right)-\tfrac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)\,du\right|\leq\tfrac{(\varrho_2-\varrho_1)^{\frac{1}{q}}}{4}\left(\tfrac{1}{q+1}\right)^{\frac{1}{q}}\left\|f'\right\|_p .
$$

Corollary 3.20 *In Theorem 3.15, if we take $x = \frac{2\varrho_1+\varrho_2}{3}$, we get the following parametrized four-point type inequalities*

$$
\begin{aligned}
&\left|\tfrac{\lambda}{3}f\left(\varrho_1\right)+\tfrac{3-2\lambda}{6}f\left(\tfrac{2\varrho_1+\varrho_2}{3}\right)+\tfrac{3-2\lambda}{6}f\left(\tfrac{\varrho_1+2\varrho_2}{3}\right)+\tfrac{\lambda}{3}f\left(\varrho_2\right)-\tfrac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)\,du\right|\\
&\leq\tfrac{(\varrho_2-\varrho_1)^{\frac{1}{q}}}{6}\left(\tfrac{1+2^{q+1}\lambda^{q+1}+2^{q+1}(1-\lambda)^{q+1}}{3(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p .
\end{aligned}
$$

Remark 3.21 By taking $\lambda = \frac{3}{8}$ in Corollary 3.20, we retrieve Corollary 9 from [3].

Corollary 3.22 *In Corollary 3.20, if we take $\lambda = \frac{39}{80}$, we get*

$$\left|\frac{1}{80}\left(13f(\varrho_1) + 27f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + 27f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + 13f(\varrho_2)\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{(\varrho_2-\varrho_1)^{\frac{1}{q}}}{240}\left(\frac{40^{q+1}+39^{q+1}+41^{q+1}}{120(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p.$$

Theorem 3.23 *Let $f:[\varrho_1,\varrho_2]\to\mathbb{R}$ be an absolutely continuous mapping on $[\varrho_1,\varrho_2]$. If f' is bounded on $[\varrho_1,\varrho_2]$, then we have*

$$\left|Q_\lambda(\varrho_1,x,\varrho_2;f) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\left(\left(\lambda^2 + (1-\lambda)^2\right)(x-\varrho_1)^2 + \left(\frac{\varrho_1+\varrho_2}{2} - x\right)^2\right)\left\|f'\right\|_\infty,$$

where $\lambda\in[0,1]$, $x\in\left[\varrho_1,\frac{\varrho_1+\varrho_2}{2}\right]$ and $Q_\lambda(\varrho_1,x,\varrho_2;f)$ is defined as (3.2).

Proof Using the absolute value in (3.1) and the fact that f' is bounded, we get

$$\left|Q_\lambda(\varrho_1,x,\varrho_2;f) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}|K(t,x)|\left|f'(t)\right|dt$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\left(\sup_{t\in[\varrho_1,\varrho_2]}\left|f'(t)\right|\right)\int_{\varrho_1}^{\varrho_2}|K(t,x)|\,dt$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{x}|t-\varrho_1-\lambda(x-\varrho_1)|\,dt\right.$$
$$\left.+\int_{x}^{\varrho_1+\varrho_2-x}\left|t-\frac{\varrho_1+\varrho_2}{2}\right|dt + \int_{\varrho_1+\varrho_2-x}^{\varrho_2}|t-\varrho_2+\lambda(x-\varrho_1)|\,dt\right)\left\|f'\right\|_\infty$$
$$= \frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{\varrho_1+\lambda(x-\varrho_1)}(\varrho_1+\lambda(x-\varrho_1)-t)\,dt + \int_{\varrho_1+\lambda(x-\varrho_1)}^{x}(t-\varrho_1-\lambda(x-\varrho_1))\,dt\right.$$
$$+\int_{x}^{\frac{\varrho_1+\varrho_2}{2}}\left(\frac{\varrho_1+\varrho_2}{2}-t\right)dt$$
$$+\int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_1+\varrho_2-x}\left(t-\frac{\varrho_1+\varrho_2}{2}\right)dt$$

$$+\int\limits_{\varrho_1+\varrho_2-x}^{\varrho_2-\lambda(x-\varrho_1)} \left(\varrho_2-\lambda\left(x-\varrho_1\right)-t\right)dt+\int\limits_{\varrho_2-\lambda(x-\varrho_1)}^{\varrho_2} \left(t-\varrho_2+\lambda\left(x-\varrho_1\right)\right)dt\Bigg)\left\|f'\right\|_\infty$$

$$=\frac{1}{\varrho_2-\varrho_1}\left(\frac{1}{2}\lambda^2\left(x-\varrho_1\right)^2+\frac{1}{2}\left(1-\lambda\right)^2\left(x-\varrho_1\right)^2+\frac{1}{2}\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2\right.$$

$$\left.+\frac{1}{2}\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2+\frac{1}{2}\left(1-\lambda\right)^2\left(x-\varrho_1\right)^2+\frac{1}{2}\lambda^2\left(x-\varrho_1\right)^2\right)\left\|f'\right\|_\infty$$

$$=\frac{1}{\varrho_2-\varrho_1}\left(\left(\lambda^2+(1-\lambda)^2\right)\left(x-\varrho_1\right)^2+\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2\right)\left\|f'\right\|_\infty.$$

The proof is completed. □

Remark 3.24 Theorem 3.23 will be reduced to

- Theorem 3 from [3], if we take $\lambda=0$,
- Corollary 3.2 from [22], if we take $x=\varrho_1$.

Corollary 3.25 *In Theorem 3.23, if we take $x=\frac{\varrho_1+\varrho_2}{2}$, we get*

$$\left|\frac{\lambda}{2}f\left(\varrho_1\right)+(1-\lambda)f\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{\lambda}{2}f\left(\varrho_2\right)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f\left(u\right)du\right|$$
$$\le\left(\frac{\lambda^2+(1-\lambda)^2}{4}\right)\left(\varrho_2-\varrho_1\right)\left\|f'\right\|_\infty.$$

Remark 3.26 Corollary 3.25 will be reduced to

- Corollary 4 from [18], if we take $\lambda=\frac{1}{3}$,
- Inequality 2.5 of Corollary 1 from [19], if we take $\lambda=\frac{1}{2}$,
- Inequality 2.1 of Remark 1 from [19], if we take $\lambda=1$.

Corollary 3.27 *In Corollary 3.25, if we take $\lambda=0$, we get*

$$\left|f\left(\frac{\varrho_1+\varrho_2}{2}\right)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f\left(u\right)du\right|\le\frac{1}{4}\left(\varrho_2-\varrho_1\right)\left\|f'\right\|_\infty.$$

Corollary 3.28 *In Theorem 3.23, if we take $x=\frac{2\varrho_1+\varrho_2}{3}$, we get*

$$\left|\frac{\lambda}{8}f\left(\varrho_1\right)+\frac{3-2\lambda}{6}f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+\frac{3-2\lambda}{6}f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+\frac{\lambda}{8}f\left(\varrho_2\right)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f\left(u\right)du\right|$$
$$\le\frac{1+4\left(\lambda^2+(1-\lambda)^2\right)}{36}\left(\varrho_2-\varrho_1\right)\left\|f'\right\|_\infty.$$

Remark 3.29 In Corollary 3.28, if we take $\lambda=\frac{3}{8}$, we get Corollary 6 from [3].

Corollary 3.30 *In Corollary 3.28, if we take $\lambda=\frac{39}{80}$, we get*

$$\left|\frac{1}{80}\left(13f\left(\varrho_1\right)+27f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+27f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+13f\left(\varrho_2\right)\right)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f\left(u\right)du\right|$$
$$\le\frac{2401}{28{,}800}\left(\varrho_2-\varrho_1\right)\left\|f'\right\|_\infty.$$

Theorem 3.31 *Let $f : [\varrho_1, \varrho_2] \to \mathbb{R}$ be a continuous differentiable on $[\varrho_1, \varrho_2]$. If $f' \in L^1[\varrho_1, \varrho_2]$, then we have*

$$\left| Q_\lambda\left(\varrho_1, x, \varrho_2; f\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\, du \right|$$
$$\leq \frac{|1-2\lambda|(x-\varrho_1)+\varrho_2-x+|(2+|1-2\lambda|)(x-\varrho_1)-(\varrho_2-x)|}{4(\varrho_2-\varrho_1)} \left\| f' \right\|_1,$$

where $\lambda \in [0,1]$, $x \in \left[\varrho_1, \frac{\varrho_1+\varrho_2}{2}\right]$ and $Q_\lambda\left(\varrho_1, x, \varrho_2; f\right)$ is defined as in (3.2).

Proof Using the absolute value in (3.1) and Hölder's inequality

$$\left| Q_\lambda\left(\varrho_1, x, \varrho_2; f\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\, du \right|$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} |K(t,x)| \left|f'(t)\right| dt$$
$$= \frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{x} |t-\varrho_1-\lambda(x-\varrho_1)| \left|f'(t)\right| dt + \int_{x}^{\varrho_1+\varrho_2-x} \left|t-\frac{\varrho_1+\varrho_2}{2}\right| \left|f'(t)\right| dt \right.$$
$$\left. + \int_{\varrho_1+\varrho_2-x}^{\varrho_2} |t-\varrho_2+\lambda(x-\varrho_1)| \left|f'(t)\right| dt \right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\varrho_2-\varrho_1}\left(\max\{\lambda(x-\varrho_1), (1-\lambda)(x-\varrho_1)\} \int_{\varrho_1}^{x} \left|f'(t)\right| dt + \left(\frac{\varrho_1+\varrho_2}{2}-x\right) \int_{x}^{\varrho_1+\varrho_2-x} \left|f'(t)\right| dt \right.$$
$$\left. + \max\{\lambda(x-\varrho_1), (1-\lambda)(x-\varrho_1)\} \int_{\varrho_1+\varrho_2-x}^{\varrho_2} \left|f'(t)\right| dt \right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\varrho_2-\varrho_1} \max\left\{\lambda(x-\varrho_1), (1-\lambda)(x-\varrho_1), \frac{\varrho_1+\varrho_2}{2}-x\right\}$$
$$\times \left(\int_{\varrho_1}^{x} \left|f'(t)\right| dt + \int_{x}^{\varrho_1+\varrho_2-x} \left|f'(t)\right| dt \int_{\varrho_1+\varrho_2-x}^{\varrho_2} \left|f'(t)\right| dt \right)$$
$$= \left(\frac{|1-2\lambda|(x-\varrho_1)+\varrho_2-x+|(2+|1-2\lambda|)(x-\varrho_1)-(\varrho_2-x)|}{4(\varrho_2-\varrho_1)} \right) \left\| f' \right\|_1.$$

This completes the proof. □

Corollary 3.32 *In Theorem 3.31, if we take $\lambda = 0$, we get*

$$\left| \frac{f(x)+f(\varrho_1+\varrho_2-x)}{2} - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\, du \right| \leq \left(\frac{(\varrho_2-\varrho_1)+4\left|x-\frac{3\varrho_1+\varrho_2}{4}\right|}{4(\varrho_2-\varrho_1)} \right) \left\| f' \right\|_1.$$

Corollary 3.33 *In Theorem 3.31, if we take $x = \varrho_1$, we get*

$$\left| \frac{f(\varrho_1)+f(\varrho_2)}{2} - \frac{1}{\varrho_2-\varrho_1} \int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \le \frac{1}{2} \left\| f' \right\|_1 .$$

Corollary 3.34 *In Theorem 3.31, if we take $x = \frac{\varrho_1+\varrho_2}{2}$, we get*

$$\left| \frac{\lambda}{2} f(\varrho_1) + (1-\lambda) f\left(\frac{\varrho_1+\varrho_2}{2}\right) + \frac{\lambda}{2} f(\varrho_2) - \frac{1}{\varrho_2-\varrho_1} \int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \le \frac{1+|1-2\lambda|}{4} \left\| f' \right\|_1 .$$

Corollary 3.35 *In Corollary 3.34, if we take $\lambda = 0$, we get*

$$\left| f\left(\frac{\varrho_1+\varrho_2}{2}\right) - \frac{1}{\varrho_2-\varrho_1} \int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \le \frac{1}{2} \left\| f' \right\|_1 .$$

Corollary 3.36 *In Corollary 3.34, if we take $\lambda = \frac{1}{3}$, we get*

$$\left| \frac{1}{6} \left(f(\varrho_1) + 4f\left(\frac{\varrho_1+\varrho_2}{2}\right) + f(\varrho_2) \right) - \frac{1}{\varrho_2-\varrho_1} \int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \le \frac{1}{3} \left\| f' \right\|_1 .$$

The same result was proven in Corollary 2.2 *from* [8].

Corollary 3.37 *In Corollary 3.34, if we take $\lambda = \frac{1}{2}$, we get*

$$\left| \frac{1}{4} \left(f(\varrho_1) + 2f\left(\frac{\varrho_1+\varrho_2}{2}\right) + f(\varrho_2) \right) - \frac{1}{\varrho_2-\varrho_1} \int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \le \frac{1}{4} \left\| f' \right\|_1 .$$

Corollary 3.38 *In Corollary 3.34, if we take $\lambda = 1$, we get*

$$\left| \frac{f(\varrho_1)+f(\varrho_2)}{2} - \frac{1}{\varrho_2-\varrho_1} \int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \le \frac{1}{2} \left\| f' \right\|_1 .$$

Corollary 3.39 *In Theorem 3.31, if we take $x = \frac{2\varrho_1+\varrho_2}{3}$, we get*

$$\left| \frac{\lambda}{3} f(\varrho_1) + \frac{3-2\lambda}{6} f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + \frac{3-2\lambda}{6} f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + \frac{\lambda}{3} f(\varrho_2) - \frac{1}{\varrho_2-\varrho_1} \int_{\varrho_1}^{\varrho_2} f(u)\,du \right|$$
$$\le \frac{1+|1-2\lambda|}{6} \left\| f' \right\|_1 .$$

Corollary 3.40 *In Corollary 3.39, if we take $\lambda = \frac{3}{8}$, we get*

$$\left| \frac{1}{8} \left(f(\varrho_1) + 3f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + 3f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + f(\varrho_2) \right) - \frac{1}{\varrho_2-\varrho_1} \int_{\varrho_1}^{\varrho_2} f(u)\,du \right| \le \frac{5}{24} \left\| f' \right\|_1 .$$

Corollary 3.41 *In Corollary 3.39, if we take $\lambda = \frac{39}{80}$, we get*

$$\left|\frac{1}{80}\left(13f\left(\varrho_1\right) + 27f\left(\frac{2\varrho_1+\varrho_2}{3}\right) + 27f\left(\frac{\varrho_1+2\varrho_2}{3}\right) + 13f\left(\varrho_2\right)\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f\left(u\right)du\right|$$
$$\leq \frac{41}{240}\left\|f'\right\|_1.$$

Theorem 3.42 *Let $f:[\varrho_1,\varrho_2]\to\mathbb{R}$ be a differentiable on $[\varrho_1,\varrho_2]$. If $f'\in L^1[\varrho_1,\varrho_2]$ and $\varkappa \leq f'(u) \leq \chi$ for all $u\in[\varrho_1,\varrho_2]$, then we have*

$$\left|Q_\lambda\left(\varrho_1,x,\varrho_2;f\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f\left(u\right)du\right| \leq \frac{(\lambda^2+(1-\lambda)^2)(x-\varrho_1)^2+\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2}{2(\varrho_2-\varrho_1)}\left(\chi-\varkappa\right),$$

where $\lambda\in[0,1]$, $x\in\left[\varrho_1,\frac{\varrho_1+\varrho_2}{2}\right]$ and $Q_\lambda\left(\varrho_1,x,\varrho_2;f\right)$ is defined as (3.2).

Proof It is easy to show that

$$\int_{\varrho_1}^{\varrho_2} K\left(t,x\right)dt = 0. \tag{3.5}$$

Let $\mathcal{C} = \frac{\varkappa+\chi}{2}$, from (3.1) and (3.5), we conclude

$$\begin{aligned}Q_\lambda\left(\varrho_1,x,\varrho_2;f\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f\left(u\right)du &= \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} K\left(t,x\right)df\left(t\right) = \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} K\left(t,x\right)f'\left(t\right)dt\\ &= \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} K\left(t,x\right)\left(f'\left(t\right)-\mathcal{C}\right)dt.\end{aligned} \tag{3.6}$$

Taking absolute value on both sides of (3.6) yields

$$\begin{aligned}&\left|Q_\lambda\left(\varrho_1,x,\varrho_2;f\right) - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f\left(u\right)du\right|\\ &\leq \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\left|K\left(t,x\right)\right|\left|f'\left(t\right)-\mathcal{C}\right|dt \leq \frac{1}{\varrho_2-\varrho_1}\max_{t\in[\varrho_1,\varrho_2]}\left|f'\left(t\right)-\mathcal{C}\right|\int_{\varrho_1}^{\varrho_2}\left|K\left(t,x\right)\right|dt.\end{aligned} \tag{3.7}$$

Using (3.7) and the fact that

$$\int_{\varrho_1}^{\varrho_2}\left|K\left(t,x\right)\right|dt = \left(\lambda^2+(1-\lambda)^2\right)\left(x-\varrho_1\right)^2 + \left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2 \tag{3.8}$$

and

$$\max_{t\in[\varrho_1,\varrho_2]}\left|f'\left(t\right)-\mathcal{C}\right| \leq \frac{\chi-\varkappa}{2}, \tag{3.9}$$

we obtain

$$\left|Q_\lambda\left(\varrho_1,x,\varrho_2;f\right)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f(u)\,du\right|\le\frac{\left(\lambda^2+(1-\lambda)^2\right)(x-\varrho_1)^2+\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2}{2(\varrho_2-\varrho_1)}\left(\chi-\varkappa\right),$$

which is the desired result. □

Remark 3.43 Theorem 3.42 will be reduced to

- Corollary 1 from [2], if we take $\lambda=0$,
- Inequality 2.12 of Corollary 2 from [30], if we take $x=\varrho_1$.

Corollary 3.44 *In Theorem 3.42, if we take $x=\frac{\varrho_1+\varrho_2}{2}$, we get*

$$\left|\frac{\lambda}{2}f(\varrho_1)+(1-\lambda)f\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{\lambda}{2}f(\varrho_2)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f(u)\,du\right|\le\frac{\left(\lambda^2+(1-\lambda)^2\right)(\varrho_2-\varrho_1)(\chi-\varkappa)}{8}.$$

Remark 3.45 Corollary 3.44 will be reduced to

- Inequality 2.11 of Corollary 1 from [30], if we take $\lambda=0$,
- Inequality 2.16 of Corollary 4 from [30], if we take $\lambda=\frac{1}{3}$,
- Inequality 2.14 of Corollary 3 from [30], if we take $\lambda=\frac{1}{2}$,
- Inequality 2.12 of Corollary 2 from [30], if we take $\lambda=1$.

Corollary 3.46 *In Theorem 3.42, if we take $x=\frac{2\varrho_1+\varrho_2}{3}$, we get*

$$\left|\frac{\lambda}{3}f(\varrho_1)+\frac{3-2\lambda}{6}f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+\frac{3-2\lambda}{6}f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+\frac{\lambda}{3}f(\varrho_2)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f(u)\,du\right|$$
$$\le\frac{\left(1+4\left(\lambda^2+(1-\lambda)^2\right)\right)(\varrho_2-\varrho_1)}{72}\left(\chi-\varkappa\right).$$

Remark 3.47 In Corollary 3.46, if we take $\lambda=\frac{3}{8}$, we get Corollary 4.3 from [1].

Corollary 3.48 *In Corollary 3.46, if we take $\lambda=\frac{39}{80}$, we get*

$$\left|\frac{1}{80}\left(13f(\varrho_1)+27f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+27f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+13f(\varrho_2)\right)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f(u)\,du\right|$$
$$\le\frac{2401(\varrho_2-\varrho_1)(\chi-\varkappa)}{57{,}600}.$$

Theorem 3.49 *Let $f:[\varrho_1,\varrho_2]\to\mathbb{R}$ be a differentiable on $[\varrho_1,\varrho_2]$. If $f'\in L^1[\varrho_1,\varrho_2]$ and $|f'|$ is convex, then we have*

$$\left|Q_\lambda\left(\varrho_1,x,\varrho_2;f\right)-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}f(u)\,du\right|$$
$$\le\frac{1}{\varrho_2-\varrho_1}\left(\frac{1-3\lambda+6\lambda^2-2\lambda^3}{6}(x-\varrho_1)^2\left(\left|f'(\varrho_1)\right|+\left|f'(\varrho_2)\right|\right)\right.$$

$$+\left(\frac{2-3\lambda+2\lambda^3}{6}\left(x-\varrho_1\right)^2+\frac{1}{2}\left(\frac{\varrho_1+\varrho_2}{2}-x\right)^2\right)\left(\left|f'\left(x\right)\right|+\left|f'\left(\varrho_1+\varrho_2-x\right)\right|\right),$$

where $\lambda\in[0,1]$, $x\in\left[\varrho_1,\frac{\varrho_1+\varrho_2}{2}\right]$ *and* $Q_\lambda\left(\varrho_1,x,\varrho_2;f\right)$ *is defined as* (3.2).

Proof From Lemma 3.1, we have

$$\begin{aligned}
&\left|Q_\lambda\left(\varrho_1,x,\varrho_2;f\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f\left(u\right)du\right|\\
&=\left|\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}K\left(t,x\right)df\left(t\right)\right|=\frac{1}{\varrho_2-\varrho_1}\left|\int_{\varrho_1}^{b}K\left(t,x\right)f'\left(t\right)dt\right|\\
&\leq\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\left|K\left(t,x\right)\right|\left|f'\left(t\right)\right|dt\\
&\leq\frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{x}\left|t-\varrho_1-\lambda\left(x-\varrho_1\right)\right|\left(\frac{t-\varrho_1}{x-\varrho_1}\left|f'\left(x\right)\right|+\frac{x-t}{x-\varrho_1}\left|f'\left(\varrho_1\right)\right|\right)dt\right.\\
&\quad+\int_{x}^{\varrho_1+\varrho_2-x}\left|t-\frac{\varrho_1+\varrho_2}{2}\right|\left(\frac{t-x}{\varrho_1+\varrho_2-2x}\left|f'\left(\varrho_1+\varrho_2-x\right)\right|+\frac{\varrho_1+\varrho_2-x-t}{\varrho_1+\varrho_2-2x}\left|f'\left(x\right)\right|\right)dt\\
&\quad\left.+\int_{\varrho_1+\varrho_2-x}^{\varrho_2}\left|t-\varrho_2+\lambda\left(x-\varrho_1\right)\right|\left(\frac{t-(\varrho_1+\varrho_2-x)}{x-\varrho_1}\left|f'\left(\varrho_2\right)\right|+\frac{\varrho_2-t}{x-\varrho_1}\left|f'\left(\varrho_1+\varrho_2-x\right)\right|\right)dt\right)\\
&=\frac{1}{\varrho_2-\varrho_1}\left(\int_{\varrho_1}^{\varrho_1+\lambda(x-\varrho_1)}\left(\varrho_1+\lambda\left(x-\varrho_1\right)-t\right)\left(\frac{t-\varrho_1}{x-\varrho_1}\left|f'\left(x\right)\right|+\frac{x-t}{x-\varrho_1}\left|f'\left(\varrho_1\right)\right|\right)dt\right.\\
&\quad+\int_{\varrho_1+\lambda(x-\varrho_1)}^{x}\left(t-\varrho_1-\lambda\left(x-\varrho_1\right)\right)\left(\frac{t-\varrho_1}{x-\varrho_1}\left|f'\left(x\right)\right|+\frac{x-t}{x-\varrho_1}\left|f'\left(\varrho_1\right)\right|\right)dt\\
&\quad+\int_{x}^{\frac{\varrho_1+\varrho_2}{2}}\left(\frac{\varrho_1+\varrho_2}{2}-t\right)\left(\frac{t-x}{\varrho_1+\varrho_2-2x}\left|f'\left(\varrho_1+\varrho_2-x\right)\right|+\frac{\varrho_1+\varrho_2-x-t}{\varrho_1+\varrho_2-2x}\left|f'\left(x\right)\right|\right)dt\\
&\quad+\int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_1+\varrho_2-x}\left(t-\frac{\varrho_1+\varrho_2}{2}\right)\left(\frac{t-x}{\varrho_1+\varrho_2-2x}\left|f'\left(\varrho_1+\varrho_2-x\right)\right|+\frac{\varrho_1+\varrho_2-x-t}{\varrho_1+\varrho_2-2x}\left|f'\left(x\right)\right|\right)dt\\
&\quad+\int_{\varrho_1+\varrho_2-x}^{\varrho_2-\lambda(x-\varrho_1)}\left(\varrho_2-\lambda\left(x-\varrho_1\right)-t\right)\left(\frac{t-(\varrho_1+\varrho_2-x)}{x-\varrho_1}\left|f'\left(\varrho_2\right)\right|+\frac{\varrho_2-t}{x-\varrho_1}\left|f'\left(\varrho_1+\varrho_2-x\right)\right|\right)dt\\
&\quad\left.+\int_{\varrho_2-\lambda(x-\varrho_1)}^{\varrho_2}\left|t-\varrho_2+\lambda\left(x-\varrho_1\right)\right|\left(\frac{t-(\varrho_1+\varrho_2-x)}{x-\varrho_1}\left|f'\left(\varrho_2\right)\right|+\frac{\varrho_2-t}{x-\varrho_1}\left|f'\left(\varrho_1+\varrho_2-x\right)\right|\right)dt\right)
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{\varrho_2-\varrho_1}\Bigg(\left|f'(\varrho_1)\right|\Bigg(\int\limits_{\varrho_1}^{\varrho_1+\lambda(x-\varrho_1)}(\varrho_1+\lambda(x-\varrho_1)-t)\tfrac{x-t}{x-\varrho_1}dt \\
&\quad+\int\limits_{\varrho_1+\lambda(x-\varrho_1)}^{x}(t-\varrho_1-\lambda(x-\varrho_1))\tfrac{x-t}{x-\varrho_1}dt\Bigg) \\
&\quad+\left|f'(x)\right|\Bigg(\int\limits_{\varrho_1}^{\varrho_1+\lambda(x-\varrho_1)}(\varrho_1+\lambda(x-\varrho_1)-t)\tfrac{t-\varrho_1}{x-\varrho_1}dt \\
&\quad+\int\limits_{\varrho_1+\lambda(x-\varrho_1)}^{x}(t-\varrho_1-\lambda(x-\varrho_1))\tfrac{t-\varrho_1}{x-\varrho_1}dt \\
&\quad+\int\limits_{x}^{\frac{\varrho_1+\varrho_2}{2}}\left(\tfrac{\varrho_1+\varrho_2}{2}-t\right)\tfrac{\varrho_1+\varrho_2-x-t}{\varrho_1+\varrho_2-2x}dt+\int\limits_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_1+\varrho_2-x}\left(t-\tfrac{\varrho_1+\varrho_2}{2}\right)\tfrac{\varrho_1+\varrho_2-x-t}{\varrho_1+\varrho_2-2x}dt\Bigg) \\
&\quad+\left|f'(\varrho_1+\varrho_2-x)\right|\Bigg(\int\limits_{x}^{\frac{\varrho_1+\varrho_2}{2}}\left(\tfrac{\varrho_1+\varrho_2}{2}-t\right)\tfrac{t-x}{\varrho_1+\varrho_2-2x}dt+\int\limits_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_1+\varrho_2-x}\left(t-\tfrac{\varrho_1+\varrho_2}{2}\right)\tfrac{t-x}{\varrho_1+\varrho_2-2x}dt \\
&\quad+\int\limits_{\varrho_1+\varrho_2-x}^{\varrho_2-\lambda(x-\varrho_1)}(\varrho_2-\lambda(x-\varrho_1)-t)\tfrac{\varrho_2-t}{x-\varrho_1}dt+\int\limits_{\varrho_2-\lambda(x-\varrho_1)}^{\varrho_2}(t-\varrho_2+\lambda(x-\varrho_1))\tfrac{\varrho_2-t}{x-\varrho_1}dt\Bigg) \\
&\quad+\left|f'(\varrho_2)\right|\Bigg(\int\limits_{\varrho_1+\varrho_2-x}^{\varrho_2-\lambda(x-\varrho_1)}(\varrho_2-\lambda(x-\varrho_1)-t)\tfrac{t-(\varrho_1+\varrho_2-x)}{x-\varrho_1}dt \\
&\quad+\int\limits_{\varrho_2-\lambda(x-\varrho_1)}^{\varrho_2}(t-\varrho_2+\lambda(x-\varrho_1))\tfrac{t-(\varrho_1+\varrho_2-x)}{x-\varrho_1}dt\Bigg)\Bigg) \\
&= \tfrac{1}{\varrho_2-\varrho_1}\Bigg(\tfrac{1-3\lambda+6\lambda^2-2\lambda^3}{6}(x-\varrho_1)^2\left(\left|f'(\varrho_1)\right|+\left|f'(\varrho_2)\right|\right) \\
&\quad+\left(\tfrac{2-3\lambda+2\lambda^3}{6}(x-\varrho_1)^2+\tfrac{1}{2}\left(\tfrac{\varrho_1+\varrho_2}{2}-x\right)^2\right)\left(\left|f'(x)\right|+\left|f'(\varrho_1+\varrho_2-x)\right|\right)\Bigg),
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\int\limits_{\varrho_1}^{\varrho_1+\lambda(x-\varrho_1)}(\varrho_1+\lambda(x-\varrho_1)-t)\tfrac{x-t}{x-\varrho_1}dt &= \int\limits_{\varrho_2-\lambda(x-\varrho_1)}^{\varrho_2}(t-\varrho_2+\lambda(x-\varrho_1))\tfrac{t-(\varrho_1+\varrho_2-x)}{x-\varrho_1}dt \\
&= \tfrac{1}{6}\left(3\lambda^2-\lambda^3\right)(x-\varrho_1)^2,
\end{aligned}
$$

$$
\begin{aligned}
\int\limits_{\varrho_1+\lambda(x-\varrho_1)}^{x}(t-\varrho_1-\lambda(x-\varrho_1))\tfrac{x-t}{x-\varrho_1}dt &= \int\limits_{\varrho_1+\varrho_2-x}^{\varrho_2-\lambda(x-\varrho_1)}(\varrho_2-\lambda(x-\varrho_1)-t)\tfrac{x-(\varrho_1+\varrho_2-t)}{x-\varrho_1}dt \\
&= \tfrac{1}{6}(1-\lambda)^3(x-\varrho_1)^2,
\end{aligned}
$$

$$\int_{\varrho_1}^{\varrho_1+\lambda(x-\varrho_1)} (\varrho_1+\lambda(x-\varrho_1)-t)\frac{t-\varrho_1}{x-\varrho_1}dt = \int_{\varrho_2-\lambda(x-\varrho_1)}^{\varrho_2} (t-\varrho_2+\lambda(x-\varrho_1))\frac{\varrho_2-t}{x-\varrho_1}dt$$
$$=\frac{1}{6}\lambda^3(x-\varrho_1)^2,$$

$$\int_{\varrho_1+\lambda(x-\varrho_1)}^{x} (t-\varrho_1-\lambda(x-\varrho_1))\frac{t-\varrho_1}{x-\varrho_1}dt = \int_{\varrho_1+\varrho_2-x}^{\varrho_2-\lambda(x-\varrho_1)} (\varrho_2-t-\lambda(x-\varrho_1))\frac{\varrho_2-t}{x-\varrho_1}dt$$
$$=\frac{1}{6}\left(2-3\lambda+\lambda^3\right)(x-\varrho_1)^2,$$

$$\int_{x}^{\frac{\varrho_1+\varrho_2}{2}} \left(\frac{\varrho_1+\varrho_2}{2}-t\right)\frac{\varrho_1+\varrho_2-x-t}{\varrho_1+\varrho_2-2x}dt = \int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_1+\varrho_2-x} \left(t-\frac{\varrho_1+\varrho_2}{2}\right)\frac{t-x}{\varrho_1+\varrho_2-2x}dt$$
$$=\frac{5}{48}(\varrho_1+\varrho_2-2x)^2$$

and

$$\int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_1+\varrho_2-x} \left(t-\frac{\varrho_1+\varrho_2}{2}\right)\frac{\varrho_1+\varrho_2-x-t}{\varrho_1+\varrho_2-2x}dt = \int_{x}^{\frac{\varrho_1+\varrho_2}{2}} \left(\frac{\varrho_1+\varrho_2}{2}-t\right)\frac{t-x}{\varrho_1+\varrho_2-2x}dt$$
$$=\frac{1}{48}(\varrho_1+\varrho_2-2x)^2.$$

The proof is completed. □

Remark 3.50 Theorem 3.42 will be reduced to

- Theorem 5 from [5], if we take $\lambda = 0$,
- Theorem 2.2 from [17], if we take $x = \varrho_1$.

Corollary 3.51 *In Theorem 3.49, if we take $x = \frac{\varrho_1+\varrho_2}{2}$, we get*

$$\left|\frac{\lambda}{2}f(\varrho_1)+(1-\lambda)f\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{\lambda}{2}f(\varrho_2)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)\,du\right|$$
$$\leq\frac{\varrho_2-\varrho_1}{24}\left(\left(1-3\lambda+6\lambda^2-2\lambda^3\right)\left(\left|f'(\varrho_1)\right|+\left|f'(\varrho_2)\right|\right)+\left(4-6\lambda+4\lambda^3\right)\left|f'\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|\right).$$

Corollary 3.52 *In Corollary 3.51, if we take $\lambda = 0$, we get*

$$\left|f\left(\frac{\varrho_1+\varrho_2}{2}\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}f(u)\,du\right|\leq\frac{\varrho_2-\varrho_1}{24}\left(\left|f'(\varrho_1)\right|+4\left|f'\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|+\left|f'(\varrho_2)\right|\right).$$

Using the convexity of $|f'|$ we get Theorem 2.2 *from* [25].

Remark 3.53 In Corollary 3.51, if we take $\lambda = \frac{1}{3}$, we recapture Corollary 2.4 from [23]. Moreover, if we use the convexity of $|f'|$, we get the third inequality of Corollary 3.4 from [31].

Corollary 3.54 *In Corollary* 3.51, *if we take* $\lambda = \frac{1}{2}$, *we get*

$$\left|\frac{1}{4}\left(f(\varrho_1)+2f\left(\frac{\varrho_1+\varrho_2}{2}\right)+f(\varrho_2)\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{\varrho_2-\varrho_1}{32}\left(\left|f'(\varrho_1)\right|+2\left|f'\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|+\left|f'(\varrho_2)\right|\right).$$

Using the convexity of $|f'|$, *we get the first inequality of Corollary* 3.4 *from* [31].

Remark 3.55 By taking $\lambda = 1$ in Corollary 3.51, we retrieve Corollary 2 from [24].

Corollary 3.56 *In Theorem* 3.49, *if we take* $x = \frac{2\varrho_1+\varrho_2}{3}$, *we get*

$$\left|\frac{\lambda}{3}f(\varrho_1)+\frac{3-2\lambda}{6}f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+\frac{3-2\lambda}{6}f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+\frac{\lambda}{3}f(\varrho_2)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{\varrho_2-\varrho_1}{216}\left(\left(4-12\lambda+24\lambda^2-8\lambda^3\right)\left(\left|f'(\varrho_1)\right|+\left|f'(\varrho_2)\right|\right)\right.$$
$$\left.+\left(11-12\lambda+8\lambda^3\right)\left(\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|+\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|\right)\right).$$

Remark 3.57 By taking $\lambda = \frac{3}{8}$ in Corollary 3.56, we retrieve Corollary 2.1 from [26].

Corollary 3.58 *In Corollary* 3.56, *if we take* $\lambda = \frac{39}{80}$, *we get*

$$\left|\frac{1}{80}\left(13f(\varrho_1)+27f\left(\frac{2\varrho_1+\varrho_2}{3}\right)+27f\left(\frac{\varrho_1+2\varrho_2}{3}\right)+13f(\varrho_2)\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} f(u)\,du\right|$$
$$\leq \frac{\varrho_2-\varrho_1}{216}\left(\frac{187{,}321}{64{,}000}\left(\left|f'(\varrho_1)\right|+\left|f'(\varrho_2)\right|\right)+\frac{388{,}919}{64{,}000}\left(\left|f'\left(\frac{2\varrho_1+\varrho_2}{3}\right)\right|+\left|f'\left(\frac{\varrho_1+2\varrho_2}{3}\right)\right|\right)\right).$$

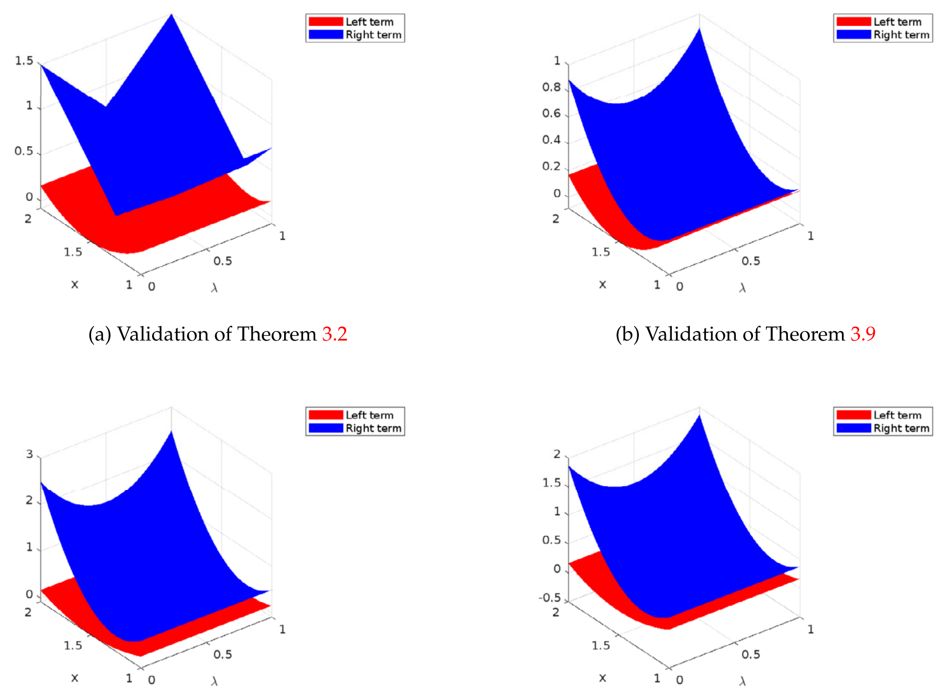
4 Illustrative example

In this section, we provide a numerical example along with some graphical representations to justify the correctness of our results.

Example 4.1 Let us consider the function f defined on the interval $[\varrho_1,\varrho_2] = [1,2]$ by $f(u) = \frac{u^2}{2}$. The derivative of this function, given by $f'(u) = u$, satisfies the conditions of most of the theorems established in this study. Indeed, we have:

- f is of bounded variation with $\bigvee_{\varrho_1}^{\varrho_2}(f) = \frac{3}{2}$, meeting the condition of Theorem 3.2.
- f is 2-Lipschitzian, satisfying the condition of Theorem 3.9.
- $\varkappa = 1 \leq f'(u) \leq 2 = \chi$, as required by Theorem 3.42.
- $|f'(u)| = u$ is convex on $[1,2]$, consistent with Theorem 3.49.

The results related to Theorems 3.2, 3.9, 3.42, and 3.49, corresponding to the function under consideration, are illustrated in Figs. 1a, 1b, 1c, and 1d, respectively.



(a) Validation of Theorem 3.2

(b) Validation of Theorem 3.9

(c) Validation of Theorem 3.42

(d) Validation of Theorem 3.49

Figure 1 Visualization of Example 4.1

From these graphical representations, it can be observed that the right-hand sides are always greater than or equal to the left-hand sides, which confirms the correctness of our results.

5 Applications

This section provides some applications of the obtained results.

5.1 Application to composite quadrature formula

Let $\mathcal{P}$ be the partition of the interval $[\varrho_1, \varrho_2]$ such that $\varrho_1 = u_0 < u_1 < \cdots < u_n = \varrho_2$, and consider the following quadrature formula:

$$\int_{\varrho_1}^{\varrho_2} f(u)\,du = \mathcal{Q}\left(f, \mathcal{P}\right) + \mathcal{E}\left(f, \mathcal{P}\right),$$

where

$$\begin{aligned}\mathcal{Q}\left(\lambda, f, \mathcal{P}\right) = &\sum_{i=0}^{n-1} \tfrac{\lambda(x_i-u_i)}{u_{i+1}-u_i} f(u_i) + \tfrac{2(1-\lambda)(x_i-u_i)+u_i+u_{i+1}-2x_i}{2(u_{i+1}-u_i)} f(x_i) \\ &+ \tfrac{2(1-\lambda)(x_i-u_i)+u_i+\varrho_2-2x_i}{2(u_{i+1}-u_i)} f(u_i+u_{i+1}-x_i) + \tfrac{\lambda(x_i-u_i)}{u_{i+1}-u_i} f(u_{i+1}),\end{aligned}$$

with $\lambda \in [0,1]$, $u_i < x_i < \frac{u_i+u_{i+1}}{2}$, and $\mathcal{E}\left(f, \mathcal{P}\right)$ denotes the associated approximation error.

Proposition 1 *Let f be as defined in Theorem 3.2, then we have*

$$\left|\mathcal{E}\left(f,\mathcal{P}\right)\right| \leq \sum_{i=0}^{n-1} \frac{(x_i-u_i)|1-2\lambda|+u_{i+1}-x_i+|(2+|1-2\lambda|)(x_i-u_i)-(u_{i+1}-x_i)|}{4} \bigvee_{u_i}^{u_{i+1}} (f)$$

Proof When we apply Theorem 3.2 to the partition $\mathcal{P}$ of subintervals $[u_i, u_{i+1}]$ ($i = 0, 1, \ldots, n-1$), we obtain

$$\left|Q_\lambda\left(u_i, x_i, u_{i+1}; f\right) - \frac{1}{u_{i+1}-u_i} \int_{u_i}^{u_{i+1}} f(u)\, du\right|$$
$$\leq \frac{(x_i-u_i)|1-2\lambda|+u_{i+1}-x_i+|(2+|1-2\lambda|)(x_i-u_i)-(u_{i+1}-x_i)|}{4(u_{i+1}-u_i)} \bigvee_{u_i}^{u_{i+1}} (f).$$

We reach the result by multiplying both sides of the aforementioned inequality by $(u_{i+1} - u_i)$, summing the generated inequalities for all $i = 0, 1, \ldots, n-1$ and applying the triangular inequality. □

5.2 Application to special means

For arbitrary real numbers φ, ψ we have:

The generalized Arithmetic mean: $A(\varrho_1, \varrho_2) = \frac{\varrho_1+\varrho_2}{2}$.

The p-Logarithmic mean: $L_p(\varrho_1, \varrho_2) = \left(\frac{b^{p+1}-\varrho_1^{p+1}}{(p+1)(\varrho_2-\varrho_1)}\right)^{\frac{1}{p}}$, $\varrho_1, \varrho_2 > 0$, $\varrho_1 \neq \varrho_2$ and $p \in \mathbb{R} \setminus \{-1, 0\}$.

Proposition 2 *Let $\varrho_1, \varrho_2 \in \mathbb{R}$ with $0 < \varrho_1 < \varrho_2$, then we have*

$$\left|A\left(\varrho_1^2, \varrho_2^2\right) + 3A^2\left(\varrho_1, \varrho_2\right) - 4L_2^2\left(\varrho_1, \varrho_2\right)\right| \leq \frac{5(\varrho_2-\varrho_1)}{4}\varrho_2.$$

Proof This follows from Theorem 3.9 by taking $x = \frac{\varrho_1+\varrho_2}{2}$ and $\lambda = \frac{1}{4}$, applied to the function $f(u) = u^2$. □

5.3 Application to probability

Proposition 3 *Consider X a random variable with the probability density function f that takes values in the finite interval $[\varrho_1, \varrho_2]$, i.e., $f : [\varrho_1, \varrho_2] \to [0, 1]$ with the cumulative distribution function $F(x) = \Pr(X \leq x) = \int_{\varrho_1}^{x} f(u)\, du$, we have*

$$\left|(1-\lambda) F\left(\frac{\varrho_1+\varrho_2}{2}\right) + \frac{\lambda}{2} - \frac{\varrho_2-E[X]}{\varrho_2-\varrho_1}\right| \leq \frac{(\lambda^2+(1-\lambda)^2)(\varrho_2-\varrho_1)(\chi-\varkappa)}{8}.$$

Proof Replace $f = F$ in Corollary 3.44 and taking into account that $F(\varrho_1) = 0$, $F(\varrho_2) = 1$ and

$$E[X] = \int_{\varrho_1}^{\varrho_2} uf(u)\, du = \varrho_2 F(\varrho_2) - \varrho_1 F(\varrho_1) - \int_{\varrho_1}^{\varrho_2} F(u)\, du = \varrho_2 - \int_{\varrho_1}^{\varrho_2} F(u)\, du.$$ □

6 Conclusion

In conclusion, this investigation sheds light on the intricacies of error estimation in the realm of four-point Newton–Cotes-type inequalities, considering a diverse array of function classes. Recognizing the dependence of error bounds on the nature of integrated functions, the study improves the applicability and reliability of numerical integration techniques. The findings not only contribute to the theoretical foundations, but also provide practical guidance for researchers and practitioners seeking optimal approaches for different function types. As numerical integration continues to play a pivotal role in scientific and engineering applications, the insights gleaned from this research pave the way for more accurate and informed decision-making in the selection of integration methods.

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Declarations

Competing interests
The authors declare no competing interests.

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