Refinement of the general form of the two-point quadrature formulas via convexity

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Abstract

In this paper, we look at the general form of the two-point quadrature formulas, which doesn't have to be symmetric. Under the convexity constraint of the first derivative, we propose a new scheme that gives the best approximation of the two-point quadrature rules.

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1. INTRODUCTION

In mathematics and science, integral inequalities provide useful tools for estimating the behavior of functions and their integrals over specified intervals. They can be used to derive estimates for the maximum and minimum values of integrals, to determine the convergence of sequences and series, and to analyze the stability of solutions to differential equations. They are also useful for solving optimization problems, where the aim is to find the function that minimizes or maximizes a given quantity. Integral inequalities have proven to be powerful tools in many areas of science and engineering, and their continued study and development will likely play a major role in advancing our understanding of complex problems in these fields.

The study of integral inequalities involves a wide range of mathematical techniques and approaches including the concept of convexity, which representes a powerful and natural property of functions that is important in many areas of pure and applied mathematics. This notion is related to the growth of the theory of inequalities, which is a key tool for studying the error estimates of numerical quadrature rules. In the recent few decades, a large number of articles addressing Newton-Cotes type inequalities through varying sorts of convexity have been published, see [Budak et al. 2021; Dragomir and Wang 1998; 1997; Hezenci et al. 2021; Iqbal et al. 2015; Kirmaci 2004; Lakhdari and Meftah 2022; Latif and Dragomir 2012] and the references therein.

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This work is focused on two-point Newton-Cotes formulas, examples of which are provided below.

In [Dragomir and Agarwal 1998], Drogomir and Agarwal look at the closed and symmetric two-point scheme below and come up with the following estimates based on the assumption that the first derivatives are convex.

Theorem 1.1. *Let $f : I \subset R \to R$ be a differentible mapping on $I, a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| \frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \frac{b-a}{8}\left(|f'(a)| + |f'(b)|\right).$$

Guessab and Schmeisser examine a general closed and symmetric two-point inequality in [Guessab and Schmeisser 2002] and establish the following estimates for Lipschitzian functions.

Theorem 1.2. *Let $f : I \to R$ be a function satisfying Lipschitz condition i.e.$|f(u) - f(v)| \le M|u-v|$, then for each $x \in \left[a, \frac{a+b}{2}\right]$ we have*

$$\left| \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le (b-a)\left[\frac{1}{8} + 2\frac{\left(x-\frac{3a+b}{4}\right)^2}{(b-a)^2}\right] M.$$

The constant $\frac{1}{8}$ is optimal in the sense that it cannot be substituted with a smaller constant.

In this note, we present a new scheme for two-point quadrature rules that provides the best approximation for the calculation of integrals. This new schema can be summarized as follows:

$$\frac{1}{b-a}\int_a^b f(u)\,du \approx \frac{\mu f(x) + \lambda f(a+b-x)}{\lambda+\mu}$$

with $x \in \left[a, \frac{a+b}{2}\right)$ and $\lambda, \mu \in R_+^*$.

2. PRELIMINARIES

Lemma 2.1. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$ and let λ, μ two positive numbers. If $f \in L^1[a,b]$, then for all $x \in \left[a, \frac{a+b}{2}\right)$ the following equality holds:*

$$\frac{\mu f(x) + \lambda f(a+b-x)}{\lambda + \mu} - \frac{1}{b-a}\int_a^b f(u)\,du = \int_0^1 K_x(t)\, f'((1-t)a + tb)\,dt, \tag{1}$$

where

$$K_x(t) = \begin{cases} (b-a)t & \text{if } t \in \left[0, \frac{x-a}{b-a}\right], \\ (b-a)\left(t - \frac{\mu}{\lambda+\mu}\right) & \text{if } t \in \left[\frac{x-a}{b-a}, \frac{b-x}{b-a}\right], \\ (b-a)(t-1) & \text{if } t \in \left[\frac{b-x}{b-a}, 1\right]. \end{cases} \tag{2}$$

PROOF. Using integration by parts, we easily obtain

$$\int_0^{\frac{x-a}{b-a}} (b-a)t f'((1-t)a+tb)\,dt = \frac{x-a}{b-a} f(x) - \frac{1}{b-a}\int_a^x f(u)\,du, \tag{3}$$

$$\begin{aligned} &\int_{\frac{x-a}{b-a}}^{\frac{b-x}{b-a}} (b-a)\left(t - \frac{\mu}{\lambda+\mu}\right) f'((1-t)a+tb)\,dt \\ =&\frac{(\lambda+\mu)(b-x) - \mu(b-a)}{(\lambda+\mu)(b-a)} f(a+b-x) - \frac{(\lambda+\mu)(x-a) - \mu(b-a)}{(\lambda+\mu)(b-a)} f(x) \qquad (4) \\ &- \frac{1}{b-a}\int_x^{a+b-x} f(u)\,du, \qquad (5) \end{aligned}$$

and

$$\int_{\frac{b-x}{b-a}}^{1} (b-a)(t-1) f'((1-t)a+tb)\,dt = \frac{x-a}{b-a} f(a+b-x) - \frac{1}{b-a}\int_{a+b-x}^{b} f(u)\,du. \tag{6}$$

Summing (3)-(6), we get the desired result.

Theorem 2.2. *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$ and let λ,μ two positive numbers. If $|f'|$ is convex, then the following inequality holds:*

$$\left|\frac{\mu f(x)+\lambda f(a+b-x)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq (b-a)\left(\left(\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2+\Pi_1\right)\left|f'(a)\right|+\left(\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2+\Pi_2\right)\left|f'(b)\right|\right),$$

where Π_1 and Π_2 are defined below.

PROOF. Using the absolute value of both sides of the expression (1) and the convexity of $|f'|$, we have

$$\left|\frac{\mu f(x)+\lambda f(a+b-x)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq\int_0^1 |K_x(t)|\left|f'((1-t)a+tb)\right|dt$$
$$\leq(b-a)\left(\int_0^{\frac{x-a}{b-a}} t\left((1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right)dt\right.$$
$$+\int_{\frac{x-a}{b-a}}^{\frac{b-x}{b-a}}\left|t-\frac{\mu}{\lambda+\mu}\right|\left((1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right)dt$$
$$\left.+\int_{\frac{b-x}{b-a}}^{1}(1-t)\left((1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right)dt\right)$$
$$=(b-a)\left(\left|f'(a)\right|\left(\int_0^{\frac{x-a}{b-a}} t(1-t)\,dt+\int_{\frac{x-a}{b-a}}^{\frac{b-x}{b-a}}\left|t-\frac{\mu}{\lambda+\mu}\right|(1-t)\,dt+\int_{\frac{b-x}{b-a}}^{1}(1-t)^2\,dt\right)\right.$$
$$\left.+\left|f'(b)\right|\left(\int_0^{\frac{x-a}{b-a}} t^2dt+\int_{\frac{x-a}{b-a}}^{\frac{b-x}{b-a}}\left|t-\frac{\mu}{\lambda+\mu}\right|t\,dt+\int_{\frac{b-x}{b-a}}^{1}(1-t)\,t\,dt\right)\right)$$
$$=(b-a)\left(\left(\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2+\Pi_1\right)\left|f'(a)\right|+\left(\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2+\Pi_2\right)\left|f'(b)\right|\right),$$

where we have used

$$\int_{0}^{\frac{x-a}{b-a}} t(1-t)\,dt = \int_{\frac{b-x}{b-a}}^{1} (1-t)\,t\,dt = \frac{1}{6}\left(\frac{3b-a-2x}{b-a}\right)\left(\frac{x-a}{b-a}\right)^{2},$$

$$\int_{0}^{\frac{x-a}{b-a}} t^{2}dt = \int_{\frac{b-x}{b-a}}^{1} (1-t)^{2}\,dt = \frac{1}{3}\left(\frac{x-a}{b-a}\right)^{3},$$

$$\begin{aligned}\Pi_1 &= \int_{\frac{x-a}{b-a}}^{\frac{b-x}{b-a}} \left|t-\frac{\mu}{\lambda+\mu}\right|(1-t)\,dt \\ &= \begin{cases} \left(\frac{\lambda-\mu}{4(\lambda+\mu)}-\frac{1}{12}\left(\frac{a+b-2x}{b-a}\right)^{2}\right)\left(\frac{a+b-2x}{b-a}\right) & \text{if } \frac{\mu}{\lambda+\mu}<\frac{x-a}{b-a} \\ \frac{2\lambda^{3}-3\lambda^{2}\mu+\mu^{3}}{12(\lambda+\mu)^{3}}+\frac{\mu}{4(\lambda+\mu)}\left(\frac{a+b-2x}{b-a}\right)^{2} & \text{if } \frac{x-a}{b-a}<\frac{\mu}{\lambda+\mu}<\frac{b-x}{b-a} \\ \left(\frac{\mu-\lambda}{4(\lambda+\mu)}+\frac{1}{12}\left(\frac{a+b-2x}{b-a}\right)^{2}\right)\left(\frac{a+b-2x}{b-a}\right) & \text{if } \frac{b-x}{b-a}<\frac{\mu}{\lambda+\mu} \end{cases}\end{aligned}$$

and

$$\begin{aligned}\Pi_2 &= \int_{\frac{x-a}{b-a}}^{\frac{b-x}{b-a}} \left|t-\frac{\mu}{\lambda+\mu}\right| t\ dt \\ &= \begin{cases} \left(\frac{\lambda-\mu}{4(\lambda+\mu)}+\frac{1}{12}\left(\frac{a+b-2x}{b-a}\right)^{2}\right)\left(\frac{a+b-2x}{b-a}\right) & \text{if } \frac{\mu}{\lambda+\mu}<\frac{x-a}{b-a} \\ \frac{\lambda^{3}-3\lambda\mu^{2}+2\mu^{3}}{12(\lambda+\mu)^{3}}+\frac{\lambda}{4(\lambda+\mu)}\left(\frac{a+b-2x}{b-a}\right)^{2} & \text{if } \frac{x-a}{b-a}<\frac{\mu}{\lambda+\mu}<\frac{b-x}{b-a} \\ \left(\frac{\mu-\lambda}{4(\lambda+\mu)}-\frac{1}{12}\left(\frac{a+b-2x}{b-a}\right)^{2}\right)\left(\frac{a+b-2x}{b-a}\right) & \text{if } \frac{b-x}{b-a}<\frac{\mu}{\lambda+\mu}. \end{cases}\end{aligned}$$

The proof is completed.

Corollary 2.3. *In Theorem 2.2, if we assume that $|f'(x)|\le M$, then we get*

$$\begin{aligned}&\left|\frac{\mu f(x)+\lambda f(a+b-x)}{\lambda+\mu}-\frac{1}{b-a}\int_{a}^{b} f(u)\,du\right| \\ &\le (b-a)M\times\begin{cases} \left(\frac{x-a}{b-a}\right)^{2}+\frac{\lambda-\mu}{2(\lambda+\mu)}\left(\frac{a+b-2x}{b-a}\right) & \text{if } \frac{\mu}{\lambda+\mu}<\frac{x-a}{b-a} \\ \frac{(\lambda-\mu)^{2}}{4(\lambda+\mu)^{2}}+\frac{1}{4}\left(\frac{a+b-2x}{b-a}\right)^{2}+\left(\frac{x-a}{b-a}\right)^{2} & \text{if } \frac{x-a}{b-a}<\frac{\mu}{\lambda+\mu}<\frac{b-x}{b-a} \\ \left(\frac{x-a}{b-a}\right)^{2}+\frac{\mu-\lambda}{2(\lambda+\mu)}\left(\frac{a+b-2x}{b-a}\right) & \text{if } \frac{b-x}{b-a}<\frac{\mu}{\lambda+\mu}. \end{cases}\end{aligned}$$

Corollary 2.4. *In Corollary 2.3, if we take $x=a$, then we obtain*

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq (b-a)\left(\frac{1}{4}+\frac{(\lambda-\mu)^2}{4(\lambda+\mu)^2}\right)M.$$

Corollary 2.5. *In Theorem 2.2, if we take $\mu=\lambda$, then we obtain*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{(x-a)^2+\left(\frac{a+b}{2}-x\right)^2}{2(b-a)}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

***Remark* 2.6.** The minimum of the above inequality is realized for $x=\frac{3a+b}{4}$.

Corollary 2.7. *In Theorem 2.2, if we take $x=a$, then we obtain*

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq (b-a)\left(\frac{2+6\left(\frac{\mu}{\lambda}\right)^2+4\left(\frac{\mu}{\lambda}\right)^3}{12\left(1+\frac{\mu}{\lambda}\right)^3}\left|f'(a)\right|+\frac{4+6\left(\frac{\mu}{\lambda}\right)+2\left(\frac{\mu}{\lambda}\right)^3}{12\left(1+\frac{\mu}{\lambda}\right)^3}\left|f'(b)\right|\right)$$
$$\leq (b-a)\left(\frac{1+2\left(\frac{\mu}{\lambda}\right)^2+\left(\frac{\mu}{\lambda}\right)^3}{2\left(1+\frac{\mu}{\lambda}\right)^3}\right)M. \tag{7}$$

Likewise, we can write

$$\left|\frac{\mu f(x)+\lambda f(a+b-x)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq (b-a)\left(\frac{2\left(\frac{\lambda}{\mu}\right)^3+6\left(\frac{\lambda}{\mu}\right)^2+4}{12\left(1+\frac{\lambda}{\mu}\right)^3}\left|f'(a)\right|+\frac{4\left(\frac{\lambda}{\mu}\right)^3+6\left(\frac{\lambda}{\mu}\right)+2}{12\left(1+\frac{\lambda}{\mu}\right)^3}\left|f'(b)\right|\right)$$
$$\leq (b-a)\left(\frac{\left(\frac{\lambda}{\mu}\right)^3+2\left(\frac{\lambda}{\mu}\right)^2+1}{2\left(1+\frac{\lambda}{\mu}\right)^3}\right)M. \tag{8}$$

***Remark* 2.8.**

—The minimum of the estimation (7) is attained when $\frac{\mu}{\lambda}=-2+\sqrt{7}$.
—The minimum of the estimation (8) is attained when $\frac{\lambda}{\mu}=-2+\sqrt{7}$.

***Remark* 2.9.** It should be noted that the best approximation is reached by one of the two following schemes

$$\int_a^b f(u)\,du \approx (b-a)\,\frac{\left(\sqrt{7}-2\right)f(x)+f(a+b-x)}{\sqrt{7}-1}$$

or

$$\int_a^b f(u)\,du \approx (b-a)\,\frac{f(x)+\left(\sqrt{7}-2\right)f(a+b-x)}{\sqrt{7}-1}.$$

Moreover, if we choose $x=a$, then we obtain the following trapezium schemes

$$\int_a^b f(u)\,du \approx (b-a)\left(\frac{\sqrt{7}-2}{\sqrt{7}-1}f(a)+\frac{1}{\sqrt{7}-1}f(b)\right)$$

or

$$\int_a^b f(u)\,du \approx (b-a)\left(\frac{1}{\sqrt{7}-1}f(a)+\frac{\sqrt{7}-2}{\sqrt{7}-1}f(b)\right).$$

3. APPLICATIONS

In this section, we'll compare the two proposals

$$TP_1=(b-a)\,\frac{\left(\sqrt{7}-2\right)f(a)+f(b)}{\sqrt{7}-1}$$

and

$$TP_2=(b-a)\,\frac{f(a)+\left(\sqrt{7}-2\right)f(b)}{\sqrt{7}-1}$$

with the classical trapezium scheme

$$TC=(b-a)\,\frac{f(a)+f(b)}{2}.$$

The most suitable scheme is chosen according to the values of $f(a)$ and $f(b)$, on the other hand, we will treat the two proposed schemes to have a better idea.

Let $f(x)=x^3$ and $[a,b]=[0,1]$. Clearly $\int_a^b f(x)\,dx=\int_0^1 x^3dx=\frac{1}{4}=0.25$,
$TC=\frac{1}{2}=0.50, TP_1=\frac{1}{\sqrt{7}-1}\simeq 0.697$ and $TP_2=\frac{\sqrt{7}-2}{\sqrt{7}-1}\simeq 0.3923$.
So, the best approximation is given by TP_2.

Let $f(x)=x^3$ and $[a,b]=[2,4]$. Clearly $\int_a^b f(x)\,dx=\int_2^4 x^3dx=60$,

$TC = 72, TP_1 = 2\frac{(\sqrt{7}-2)8+64}{\sqrt{7}-1} \simeq 84,34$ and $TP_2 = 2\frac{8+(\sqrt{7}-2)64}{\sqrt{7}-1} \simeq 59,94.$
So, the best approximation is given by TP_2.
Let $f(x) = \frac{1}{x}$ and $[a,b] = [1,2]$. Clearly $\int\limits_a^b f(x)\,dx = \int\limits_1^2 \frac{1}{x}dx = \ln 2 \simeq 0.6931,$
$TC = \frac{3}{4} = 0.75, TP_1 = \frac{(\sqrt{7}-2)+0.5}{\sqrt{7}-1} \simeq 0.6951$ and $TP_2 = \frac{1+(\sqrt{7}-2)0.5}{\sqrt{7}-1} \simeq 0.8048.$
So, the best approximation is given by TP_1.

4. CONCLUSION

In the classical trapezium rule $f(a)$ and $f(b)$ represent graphically the small and large base of a rectangular trapezium. In the proposed diagram, the large base will get smaller and the small base will get bigger, or the opposite will happen. This means that the angled side makes a small rotation around the point $\left(\frac{a+b}{2}, f(\frac{a+b}{2})\right)$, which is a fixed point for all angled sides. We end up with several trapezoids, and we can figure out the area of the one closest to the curve's surface by multiplying the large base by $\frac{2(\sqrt{7}-2)}{\sqrt{7}-1}$.

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REFERENCES

BUDAK, H., HEZENCI, F., AND KARA, H. 2021. On parameterized inequalities of ostrowski and simpson type for convex functions via generalized fractional integrals. *Mathematical Methods in the Applied Sciences 44,* 17, 12522–12536.

DRAGOMIR, S. AND AGARWAL, R. 1998. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Applied mathematics letters 11,* 5, 91–95.

DRAGOMIR, S. AND WANG, S. 1997. An inequality of ostrowski-grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules. *Computers & Mathematics with Applications 33,* 11, 15–20.

DRAGOMIR, S. AND WANG, S. 1998. Applications of ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules. *Applied Mathematics Letters 11,* 1, 105–109.

GUESSAB, A. AND SCHMEISSER, G. 2002. Sharp integral inequalities of the hermite–hadamard type. *Journal of approximation theory 115,* 2, 260–288.

HEZENCI, F., BUDAK, H., AND KARA, H. 2021. New version of fractional simpson type inequalities for twice differentiable functions. *Advances in Difference Equations 2021*, 1–10.

IQBAL, M., IQBAL BHATTI, M., AND NAZEER, K. 2015. Generalization of inequalities analogous to hermite-hadamard inequality via fractional integrals. *Bulletin of the Korean Mathematical Society 52,* 3, 707–716.

KIRMACI, U. S. 2004. Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Applied mathematics and computation 147,* 1, 137–146.

LAKHDARI, A. AND MEFTAH, B. 2022. Some fractional weighted trapezoid type inequalities for preinvex functions. *International Journal of Nonlinear Analysis and Applications 13,* 1, 3567–3587.

LATIF, M. AND DRAGOMIR, S. S. 2012. On some new inequalities for differentiable co-ordinated convex functions. *Journal of Inequalities and Applications 2012,* 1, 1–13.

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